LARGEVARS: AN R PACKAGE FOR TESTING LARGE VARS FOR THE PRESENCE OF COINTEGRATION

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Abstract. Cointegration is a property of multivariate time series that determines whether its non-stationary, growing components have a stationary linear combination. **Largevars** R package conducts a cointegration test for high-dimensional vector autoregressions of order k based on the large N, T asymptotics of Bykhovskaya and Gorin [2022, 2025]. The implemented test is a modification of the Johansen likelihood ratio test. In the absence of cointegration the test converges to the partial sum of the Airy_1 point process, an object arising in random matrix theory.

The package and this article contain simulated quantiles of the first ten partial sums of the Airy_1 point process that are precise up to the first 3 digits. We also include two examples using **Largevars**: an empirical example on S&P100 stocks and a simulated VAR(2) example.

1. Introduction

Vector Autoregressions (VARs) are a fundamental tool in econometrics and time series analysis, providing a framework for modeling the dynamic interrelationships among multiple time series. However, as the number of variables in a VAR increases, the complexity of the model grows significantly, posing challenges for both estimation and inference. One critical aspect of analyzing VAR models is testing for the presence of cointegration, which can inform whether a set of non-stationary series share a long-run equilibrium relationship. That is, whether a set of non-stationary time series has a stationary linear combination, see, e.g., Johansen [1995].

There are several ways to test for the presence of cointegration (see, e.g., Maddala and Kim [1998] for the detailed description of various methods). Yet traditional tests for the presence of cointegration (e.g., likelihood ratio of Johansen [1988, 1991]) are not suitable for analyzing large systems, as they tend to significantly over-reject the null hypothesis, see, for example, Ho and Sørensen [1996], Gonzalo and Pitarakis [1999]. To address this issue, Bykhovskaya and Gorin [2022, 2025] propose an approach based on alternative asymptotics, where both the number of coordinates N and the length T of time series are large, and tailored for high-dimensional time series. **Largevars** package implements this approach.

The test implemented in the **Largevars** package is based on the squared sample canonical correlations between transformed past levels (lags) and changes (first differences) of the data, as outlined in Section 2.2. Its asymptotic distribution (derived under $N, T \to \infty$ jointly and proportionally) is given by the partial sums of the Airy_1 point process, a random matrix object defined in Section 2.3. The quantiles of the sums are necessary in order to implement the test; we have tabulated them and included both in the package and in the tables presented in this article. These tables are of independent interest and, with the exception of Table 1, have not appeared in the literature before. Table 1 corresponds to the quantiles of the Tracy-Widom distribution, which were also tabulated in Bejan [2005].

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Another R package that provides quantiles of random matrix origin is **RMTstat** [Johnstone et al., 2022], which offers density, distribution, and quantile functions for the Tracy–Widom distribution with parameters $\beta = 1, 2, 4$, based on precomputed tables. In comparison, our package provides Tracy-Widom quantiles for $\beta = 1$ in Table 1, along with nine additional tables containing quantiles necessary for cointegration testing based on the Airy_1 partial sums. The method used in **RMTstat** to generate quantile tables is not applicable in our more general setting. Our alternative nontrivial algorithm, along with its MATLAB implementation, is described in detail in Section 2.3.

Various R packages have been developed for cointegration testing. For instance, the **urca** package Pfaff [2008] includes functions such as `ca.jo` (implementing the procedure of Johansen [1988, 1991]), `ca.po` (for the test of Phillips and Ouliaris [1990]), and `cajolst` (implementing the procedure of Lütkepohl et al. [2004]). The **ARDL** package Natsiopoulos and Tzeremes [2023] provides functions `bounds_f_test` and `bounds_t_test`, which implement the Wald bounds-test and t-bounds test for no cointegration by Pesaran et al. [2001]. Additionally, the **bootCT** package Vacca and Bertelli [2024] offers `boot_ardl` function for the bootstrap version of ARDL cointegration tests, as in Bertelli et al. [2022]. Several other implementations of cointegration tests exist in other software languages. Despite these offerings, the majority of existing packages are not tailored for high-dimensional settings, and their performance on time-series with large N remains uncertain (see Onatski and Wang [2018, 2019] for detailed theoretical discussions on over-rejection in classical tests for large N). Some of the packages explicitly prohibit the use of large N, e.g. `ca.jo` in the **urca** package does not output test results for $N > 10$. Therefore, our new package complements existing software by providing specialized tools with theoretical assurances for high-dimensional settings.

2. Cointegration test

This section explains the theoretical foundations of the cointegration testing and challenges in their practical implementation.

2.1. Setup and likelihood ratio test.

We consider a $N \times (T+1)$ data set represented by columns X_t, $0 \le t \le T$. These columns are interpreted as observations of an N-dimensional vector at $T+1$ time points or as N scalar time series. X_t is said to be *cointegrated* if there exists a linear combination with coefficients β of these time series such that the scalar time series $\beta^\top X_t$, for $0 \le t \le T$, is *stationary* over time, potentially after detrending. Conversely, if every linear combination is non-stationary, we conclude no cointegration exists. In stationary scenarios, there is no growth over time, and correlations exhibit short-range behavior. Non-stationary scenarios, however, show growth and wider correlations, see Brockwell and Davis [1991], Johansen [1995] for rigorous treatments. In particular, when $N = 2$, cointegration implies a long-term equilibrium where the first and second coordinates of X_t move together.

Cointegration has been extensively studied in econometrics, beginning with seminal works by Granger [1981], Engle and Granger [1987]. Many variables in macroeconomics and finance, such as price levels, consumption, output, trade flows, and interest rates, may exhibit cointegration. A classic example is the relationship between interest rates for 3-month and 1-year US Treasury bills, which are cointegrated (see, e.g., Bykhovskaya and Gorin [2024, Section 5.2] for an illustration). In portfolio management cointegrated stocks give rise to the strategy known as "pairs trading". This article and its accompanying software discuss statistical procedures for determining whether a given data set X_t demonstrates cointegration.

The main advantage of our approach is its applicability to settings with large N, in contrast to many existing packages and articles.

For the mathematical setup we assume that the data set is a realization of an N-dimensional vector autoregressive process of order k, denoted VAR(k). The process is driven by a sequence of i.i.d. mean-zero errors ε_t with non-degenerate covariance matrix Λ. In its error correction form, the model reads:

$$\Delta X_t = \mu + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \Pi X_{t-k} + \varepsilon_t, \qquad t = 1, \dots, T, \tag{1}$$

where $\Delta X_t := X_t - X_{t-1}$. The parameters $\mu \in \mathbb{R}^N$, and $\Gamma_1, \dots, \Gamma_{k-1}, \Pi \in \mathbb{R}^{N \times N}$ are unknown. The process is initialized with fixed values $X_{1-k}, \dots, X_0$.

(Some authors use a slightly different form: $\Delta X_t = \mu + \Pi X_{t-1} + \sum_{i=1}^{k-1} \tilde{\Gamma}_i \Delta X_{t-i} + \varepsilon_t$, but the distinction is not crucial for our discussion.)

It is well known (see Engle and Granger [1987], Johansen [1995]) that, under technical conditions, the process X_t is cointegrated if and only if $\Pi \neq 0$. In particular, testing for the absence of cointegration can be recast as testing the hypothesis $\Pi = 0$ in model (1).

A widely used procedure for testing this hypothesis is the Johansen likelihood ratio test, introduced in Johansen [1988, 1991] and based on Gaussian maximum likelihood (see also Anderson [1951]). The method involves computing the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ of a certain matrix constructed from the observed data X_t. These eigenvalues lie in the interval $[0, 1]$ and can be interpreted as squared sample canonical correlations between transformed lagged levels and first differences of the time series. We describe a slightly modified version of this procedure in Section 2.2.

The statistic for testing the hypothesis

$$H_0 : \ \operatorname{rank}(\Pi) = 0 \qquad (\text{i.e., } \Pi \equiv 0)$$

against the alternative

$$H(r) : \quad \operatorname{rank}(\Pi) \in [1, r]$$

is based on the r largest eigenvalues and takes the form:

$$\sum_{i=1}^{r} \ln(1 - \lambda_i).$$

Under H_0 the λ_i's tend to be small, making the statistic close to zero. Under the alternative $H(r)$, one expects some λ_i to be close to one, which makes the sum more negative. Thus, the test rejects H_0 when the statistic is sufficiently negative.

The parameter r is user-specified, and for fixed N this gives rise to $N-1$ different tests corresponding to $r = 1, 2, \dots, N-1$. The asymptotic distribution of the statistic $\sum_{i=1}^{r} \ln(1-\lambda_i)$ under H_0 as $T \to \infty$, with N fixed was derived in Johansen [1988, 1991]. It involves the eigenvalues of a certain matrix of Itô integrals. This result underpins the classical cointegration testing procedure: compute the test statistic from data and compare it to the quantiles of its theoretical asymptotic distribution under H_0; if the observed value is smaller than the critical threshold, reject H_0 and conclude that the series X_t exhibits cointegration.

In practice the Johansen procedure and its associated software are typically applied only when N is small. The main reason is that the quality of the approximation based on the asymptotic distribution involving Itô integrals deteriorates rapidly as N increases. Specifically, the distribution of the statistic $\sum_{i=1}^{r} \ln(1 - \lambda_i)$ can deviate significantly from its theoretical

large-T limit even for moderate values of N (for example, the case $N = 10$, $T = 100$ already yields poor performance). As a result, the likelihood ratio test tends to over-reject the null hypothesis H_0 in finite samples. See Onatski and Wang [2018, 2019] for a detailed theoretical analysis of this phenomenon.

This breakdown highlights the need to adapt the testing procedure and software for high-dimensional settings. In this work, we address this challenge by developing a new approach that is reliable for large N, building on recent theoretical results from Bykhovskaya and Gorin [2022, 2025].

2.2. **The procedure adapted for large N.** We now discuss a procedure implemented in **Largevars** package, which is a modification of the construction of eigenvalues λ_i used in the Johansen likelihood test. Starting with the data set X_t, $0 \le t \le T$, we fix a number $r = 1, 2, \dots$ and perform the following steps.

Step 1 (Detrending). We de-trend and shift the data by defining

$$\tilde{X}_t = X_{t-1} - \frac{t-1}{T}(X_T - X_0), \qquad 1 \le t \le T. \tag{2}$$

Step 2 (Cyclic indexing and regressor construction). We define cyclic indices modulo T: for any $a \in \mathbb{Z}$, set

$$a \mid T = a + mT, \quad \text{where } m \in \mathbb{Z} \text{ is such that } a + mT \in \{1, 2, \dots, T\}.$$

Using this notation, we construct the following regressor matrices:

$$\tilde{Z}_{0t} = \Delta X_{t|T} \equiv \Delta X_t, \quad \tilde{Z}_{kt} = \tilde{X}_{t-k+1|T}, \quad \tilde{Z}_{1t} = (\Delta X_{t-1|T}^\top, \dots, \Delta X_{t-k+1|T}^\top, 1)^\top, \quad 1 \le t \le T.$$

Here and below $^\top$ denotes matrix transposition. For each fixed t, the vector $\tilde{Z}_{1t}$ is a column of dimension $((k-1)N + 1) \times 1$. The index k in $\tilde{Z}_{kt}$ is used symbolically to reflect the VAR(k) structure and does not refer to a specific numerical value — this convention follows Johansen [1988, 1991].

We emphasize that due to the use of cyclic indices, values of X_t at $t = 0, -1, \dots$ are replaced by values at $t = T, T-1, \dots$. However, when $k = 1$ (i.e., for a VAR(1) model), no negative indices arise for $1 \le t \le T$, and the cyclic indexing becomes irrelevant.

Step 3 (Regression residuals). We compute the residuals from the regressions of $\tilde{Z}_{0t}$ and $\tilde{Z}_{kt}$ on $\tilde{Z}_{1t}$:

$$\tilde{R}_{it} = \tilde{Z}_{it} - \left(\sum_{\tau=1}^{T} \tilde{Z}_{i\tau}\tilde{Z}_{1\tau}^\top\right)\left(\sum_{\tau=1}^{T} \tilde{Z}_{1\tau}\tilde{Z}_{1\tau}^\top\right)^{-1} \tilde{Z}_{1t}, \qquad i = 0, k. \tag{3}$$

Step 4 (Canonical Correlations). Let $\tilde{R}_i$ be the $N \times T$ matrix whose columns are $\tilde{R}_{it}$ for $1 \le t \le T$, for $i = 0, k$. Define the cross-product matrices

$$\tilde{S}_{ij} = \sum_{t=1}^{T} \tilde{R}_{it}\tilde{R}_{jt}^*, \qquad i, j \in \{0, k\}, \tag{4}$$

and form the matrix

$$\tilde{\mathcal{C}} = \tilde{S}_{k0}\tilde{S}_{00}^{-1}\tilde{S}_{0k}\tilde{S}_{kk}^{-1}. \tag{5}$$

The eigenvalues $\tilde{\lambda}_1 \ge \dots \ge \tilde{\lambda}_N$ of $\tilde{\mathcal{C}}$ represent the squared sample canonical correlations between $\tilde{R}_k$ and $\tilde{R}_0$. Equivalently, they solve the eigenvalue problem

$$\det\left(\tilde{S}_{k0}\tilde{S}_{00}^{-1}\tilde{S}_{0k} - \tilde{\lambda}\tilde{S}_{kk}\right) = 0. \tag{6}$$

Step 5 (Test Statistic). We construct the modified likelihood ratio statistic:

$$LR_{N,T}(r) = \sum_{i=1}^{r} \ln(1 - \tilde{\lambda}_i). \tag{7}$$

The subscript (N, T) indicates that this version of the Johansen LR test is tailored for the high-dimensional regime where both N and T are large. After centering and scaling, the statistic $LR_{N,T}(r)$ is compared with suitable critical values to decide whether to reject the null hypothesis H_0. Heuristically, rejection corresponds to the case where the largest eigenvalues $\tilde{\lambda}_i$ are significantly large and well-separated from the rest.

We now describe the asymptotic distribution theory underlying the critical values used in our procedure. These formulas are based on the limiting behavior of the test statistic $LR_{N,T}(r)$ as both $N, T \to \infty$. The relevant asymptotics were developed in Bykhovskaya and Gorin [2022, 2025], and we briefly recall the key results.

The limiting distribution involves a stochastic object known as the *Airy*$_1$ *point process*, denoted by $\{\mathfrak{a}_i\}_{i=1}^{\infty}$. This is a random, strictly decreasing sequence of real numbers: $\mathfrak{a}_1 > \mathfrak{a}_2 > \mathfrak{a}_3 > \dots$. We discuss this process in more detail in the next section.

Let T, N, and k be such that $\frac{T}{N} > k + 1$. Define the following constants:

$$\mathfrak{p} = 2, \qquad \mathfrak{q} = \frac{T}{N} - k, \qquad \lambda_{\pm} = \frac{1}{(\mathfrak{p}+\mathfrak{q})^2}\left[\sqrt{\mathfrak{p}(\mathfrak{p}+\mathfrak{q}-1)} \pm \sqrt{\mathfrak{q}}\right]^2, \tag{8}$$

$$c_1(N,T) = \ln(1-\lambda_+), \qquad c_2(N,T) = -\frac{2^{2/3}\lambda_+^{2/3}}{(1-\lambda_+)^{1/3}(\lambda_+-\lambda_-)^{1/3}}(\mathfrak{p}+\mathfrak{q})^{-2/3} < 0.$$

Then, under appropriate assumptions, it follows from Bykhovskaya and Gorin [2022, Theorem 2] and Bykhovskaya and Gorin [2025, Theorem 9] that

$$\frac{\sum_{i=1}^{r} \ln(1-\tilde{\lambda}_i) - r \cdot c_1(N,T)}{N^{-2/3} c_2(N,T)} \xrightarrow{d} \sum_{i=1}^{r} \mathfrak{a}_i, \qquad N, T \to \infty. \tag{9}$$

The practical applicability of this result depends on whether the theoretical assumptions hold for a given data set X_t. In Section 2.4 we discuss model diagnostics that users can perform to assess the validity of the asymptotic approximation in real data.

To carry out the test in practice, one starts with the statistic $LR_{N,T}(r)$ defined in (7). We recommend choosing small values of r (e.g., $r = 1, 2$, or 3). The approximation in (9) assumes that r is fixed as $N, T \to \infty$ — the rationale for this and its implications are discussed in detail in Bykhovskaya and Gorin [2022, Section 3.2].

The testing procedure then proceeds by computing the rescaled statistic:

$$\frac{LR_{N,T}(r) - r \cdot c_1(N,T)}{N^{-2/3} c_2(N,T)}$$

and comparing it to the quantiles of the distribution $\sum_{i=1}^{r} \mathfrak{a}_i$. If the rescaled value exceeds the α-quantile, we reject the null hypothesis of no cointegration at the $(1-\alpha)$ significance level. The function `largevar()` in the **Largevars** package implements this procedure.

2.3. **Simulation of the Airy$_1$ point process.** The asymptotic formula (9) shows that implementing our cointegration testing procedure requires knowledge of the distribution of the random variables $\sum_{i=1}^{r} \mathfrak{a}_i$. In this section we discuss how this distribution can be computed.

The Airy$_1$ point process is a random infinite sequence of real numbers $\mathfrak{a}_1 > \mathfrak{a}_2 > \mathfrak{a}_3 > \dots$ that can be defined via the following proposition:

Proposition 1 (Forrester [1993], Tracy and Widom [1996])**.** *Let Y_N be an $N \times N$ matrix of i.i.d. $\mathcal{N}(0,2)$ Gaussian random variables, and let $\mu_{1;N} \geq \mu_{2;N} \geq \dots \mu_{N;N}$ be eigenvalues of $\frac{1}{2}\left(Y_N + Y_N^\top\right)$. Then, in the sense of convergence of finite-dimensional distributions,*

$$\lim_{N\to\infty} \left\{ N^{1/6}\left(\mu_{i;N} - 2\sqrt{N}\right)\right\}_{i=1}^{N} = \{\mathfrak{a}_i\}_{i=1}^{\infty}. \tag{10}$$

The distribution of $\mathfrak{a}_1$ is known as the Tracy–Widom distribution F_1. Tracy and Widom [1996] showed that the cumulative distribution function of F_1 can be expressed as the solution to a Painlevé differential equation. Numerical solutions to this equation were used by Bejan [2005] to compute highly accurate tables of quantiles of F_1. See also Dieng [2005], Bornemann [2010], Trogdon and Zhang [2024] for further numerical advances.

Several software packages incorporate precomputed tables of the Tracy–Widom distribution for practical use. For example, the **RMTstat** package in R provides such functionality; see Johnstone et al. [2022].

When $r > 1$, much less is known about the distribution of $\sum_{i=1}^{r} \mathfrak{a}_i$, and it is unclear whether any of the approaches described in the previous paragraph remain applicable. Therefore, in the **Largevars** package, we embed precomputed quantile tables for $\sum_{i=1}^{r} \mathfrak{a}_i$ obtained through direct simulation based on the definition in (10).

The convergence rate in (10) is of order $N^{-1/3}$, which is relatively slow. For instance, even with a large matrix of size $N = 1000$, the approximation error remains around $N^{-1/3} = 0.1$. Johnstone and Ma [2012] proposed computational techniques that accelerate convergence to $N^{-2/3}$ for $\mathfrak{a}_1$, but it is unknown whether such techniques yield similar improvements for $\sum_{i=1}^{r} \mathfrak{a}_i$ with $r > 1$. Moreover, testing this is nontrivial: while Johnstone and Ma [2012] could compare their numerics against the known distribution of $\mathfrak{a}_1$, no such benchmark exists for the sum of the top r points. Consequently, to ensure accurate quantile estimation via (10), we opted for a very large matrix size: $N = 10^8$.

Using (10) with $N = 10^8$ presents a computational challenge: no modern system can compute the eigenvalues of a $10^8 \times 10^8$ dense matrix. To circumvent this, we employ a numerical technique based on the tridiagonalization of the symmetric matrix $\frac{1}{2}(Y_N + Y_N^\top)$, following the approach of Dumitriu and Edelman [2002]. This reduces the problem to computing the eigenvalues of a real symmetric tridiagonal matrix of size $N \times N$:

$$\begin{pmatrix} \mathcal{N}(0,2) & \chi_{N-1} & 0 & & & 0 \\ \chi_{N-1} & \mathcal{N}(0,2) & \chi_{N-2} & & & \\ 0 & \chi_{N-2} & \mathcal{N}(0,2) & & & \\ & & & \ddots & & \\ & & & & \mathcal{N}(0,2) & \chi_1 \\ 0 & & & & \chi_1 & \mathcal{N}(0,2) \end{pmatrix}, \tag{11}$$

where all entries on or above the diagonal are independent. Here, $\mathcal{N}(0,2)$ denotes a normal random variable with mean 0 and variance 2, and χ_ℓ denotes the square root of a chi-squared random variable with ℓ degrees of freedom.

Directly computing the eigenvalues of the full matrix in (11) for $N = 10^8$ remains computationally infeasible. However, one can instead consider the eigenvalues of its top-left $\sqrt{N} \times \sqrt{N}$ submatrix. Owing to the specific structure of (11), the largest eigenvalues of the full $N \times N$ matrix and those of the $\sqrt{N} \times \sqrt{N}$ submatrix have the same asymptotic distribution; see Edelman and Persson [2005, Section 1.1] and Johnstone et al. [2021, Lemma 5.2] for theoretical justification.

We leverage this result in our simulations by performing 10^7 Monte Carlo runs on symmetric tridiagonal random matrices of size $10^4 \times 10^4$, corresponding to the top-left corner of the tridiagonal matrix in (11) with $N = 10^8$. After computing their eigenvalues, we rescale them according to the transformation in (10) with $N = 10^8$. This yields approximations for the distribution of the individual $\mathfrak{a}_i$ values and, consequently, for the sums $\sum_{i=1}^{r} \mathfrak{a}_i$. We carried out this procedure for $r = 1, 2, \ldots, 10$.

To assess the quality of our approximation, we do not run all 10^7 simulations in a single batch. Instead, we perform 10^6 simulations ten times using different initial random seeds. The average of the resulting sample quantiles provides our estimates for the quantiles of $\sum_{i=1}^{r} \mathfrak{a}_i$, while the standard deviation across the ten runs (not shown here) offers a measure of their reliability. The resulting quantile tables are embedded within the package and also presented separately in Section 5. The standard deviations suggest that the error is at most ± 1 in the third significant digit, i.e., ± 0.01 for $r = 1$ and ± 0.1 for $r = 10$. For $r = 1$, our results closely match those of Bejan [2005].

The simulation scripts were written in MATLAB and executed on a computing cluster provided by the Department of Economics at Duke University. Due to the substantial runtime, these simulations cannot be executed "on the fly" within the R package and also it limits our ability to further increase the precision of the quantile tables. For reproducibility, we include both the full script and a reduced version (with smaller N and a fixed random seed), along with its output — a low-precision version of the tables presented in Section 5.

Finding higher-precision quantiles for $\sum_{i=1}^{r} \mathfrak{a}_i$ remains an open problem, whether by theoretical or numerical means. We hope that the tables in Section 5 will help stimulate further interest in this question.

2.4. **Model fit assessment.** The procedure for cointegration testing relies on the validity of the approximation (9) under the null hypothesis H_0 of no cointegration. This approximation, in turn, depends on theoretical assumptions made in its derivation. A natural concern for users is whether these assumptions are reasonable for a given dataset X_t, $0 \le t \le T$.

Bykhovskaya and Gorin [2025, Theorem 9] provides a mathematical justification for (9) under the setting where both T and N are large, with $T/N \in (k+1, \infty)$ bounded away from the endpoints. The proof assumes Gaussian errors ε_t in (1) and imposes the restriction $\Gamma_1 = \Gamma_2 = \cdots = \Gamma_{k-1} = 0$. The discussion following the theorem in the same article argues that the approximation should remain valid when the Γ_i matrices are of low rank, and Bykhovskaya and Gorin [2022, Section 7.1] further argues that the Gaussianity assumption on ε_t is likely not essential.

Moreover, Onatski and Wang [2018] and Bykhovskaya and Gorin [2025, Theorem 3] demonstrate that under these relaxed assumptions an additional result holds — independent of whether cointegration is present (i.e., whether $\Pi = 0$ in (1) or not), as long as Π remains of small rank. Specifically, they show that the histogram of *all* eigenvalues $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_N$ from (6) converges to a deterministic limiting distribution.

In more detail, the Wachter distribution is a probability distribution on the interval $[0,1]$ that depends on two parameters $\mathfrak{p} > 1$ and $\mathfrak{q} > 1$, and is defined by the density

$$\mu_{\mathfrak{p},\mathfrak{q}}(x) = \frac{\mathfrak{p}+\mathfrak{q}}{2\pi} \cdot \frac{\sqrt{(x-\lambda_-)(\lambda_+ - x)}}{x(1-x)} \mathbf{1}_{[\lambda_-,\lambda_+]}, \tag{12}$$

where the support $[\lambda_-, \lambda_+] \subset (0,1)$ is given by

$$\lambda_\pm = \frac{1}{(\mathfrak{p}+\mathfrak{q})^2} \left(\sqrt{\mathfrak{p}(\mathfrak{p}+\mathfrak{q}-1)} \pm \sqrt{\mathfrak{q}} \right)^2. \tag{13}$$

Onatski and Wang [2018] and Bykhovskaya and Gorin [2025] show that if the parameters are chosen according to (8), then the empirical distribution of the eigenvalues $\tilde{\lambda}_i$ from (6) converges to the Wachter distribution:

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{\tilde{\lambda}_i} \longrightarrow \mu_{\mathfrak{p},\mathfrak{q}}(x)\,\mathrm{d}x, \qquad N,T \to \infty, \quad \text{weakly, in probability.} \tag{14}$$

This convergence provides a practical check for the applicability of our procedure. If, for a given data set X_t, the histogram of eigenvalues $\tilde{\lambda}_i$ resembles the shape of the Wachter distribution (possibly excluding a few outlying values, which may correspond to cointegration), then it is reasonable to trust the assumptions behind our cointegration test. If not, the modeling assumptions are likely violated, and the test results should not be used.

To facilitate this diagnostic step, the function `largevar()` in our package includes an option to plot the empirical histogram of eigenvalues along with the Wachter density.

3. Commands

3.1. **Getting started.** The latest version of the **Largevars** package can always be found on Github and installed using the `devtools` R package:

```
library(devtools)
install_github("eszter-kiss/Largevars")
```

The latest stable version from CRAN can be installed by `install.packages("Largevars")`. Help for using the functions in the package can be called by running ?? `function name` . The empirical example in Section 4.1 of this paper can provide further guidance.

3.2. **Function** `largevar.` `largevar()` is the main function in the package that implements the cointegration test for high-dimensional VARs.

```
largevar(data, k = 1, r = 1,  fin_sample_corr = FALSE,  plot_output = TRUE,
  significance_level = 0.05)
```

`data` — A numeric matrix where the columns contain individual time series that will be examined for the presence of cointegrating relationships. The rows are indexed by $t = 0, 1, \ldots, T$ and the columns by $i = 1, \ldots, N$. In the notations of Section 2, this is X_t, $0 \le t \le T$.

`k` — The number of lags that we wish to employ in the vector autoregression, as in (1). The default value is `k = 1`.

`r` The number of largest eigenvalues used in the test as in (7). The default value is `r = 1`.

`fin_sample_corr` A boolean variable indicating whether we wish to employ finite sample correction on our test statistic, as suggested in Bykhovskaya and Gorin [2022, Discussion after Theorem 2 and Section 5.1], Bykhovskaya and Gorin [2025, Footnote 13]. The default value is `fin_sample_corr = FALSE`.

`plot_output` A boolean variable indicating whether we wish to generate a plot of the empirical distribution of eigenvalues discussed in Section 2.4. The default value is `plot_output = TRUE`.

`significance_level` Specify the significance level at which the decision about the H_0 should be made. This is denoted $(1-\alpha)$ in Section 2.2. The default value is `significance_level = 0.05`.

The function `largevar()` operates according to the steps laid in out in Section 2.2. The test statistic is formed based on the r largest eigenvalues. Any value of r can be used to reject the hypothesis H_0 of no cointegration, and the user can try different options. We recommend small values such as $r = 1, 2, 3$, see Bykhovskaya and Gorin [2022, Section 3.2] for the detailed discussion.

`largevar()` returns a list object that contains the test statistic, a statistical table with a subset of theoretical quantiles ($q = 0.90, 0.95, 0.97, 0.99$) presented for $r = 1$ to $r = 10$, the decision about H_0 at the significance level specified by the user, and the p-value. These can be accessed by `list$statistic` (numeric value) `list$significance_test$significance_table` (numeric matrix), `list$significance_test$boolean_decision` (numeric value of 0 or 1, where 1 means "reject"), and `list$significance_test$p_value` (numeric value), respectively.

The simulations for the quantiles of the limiting distribution were conducted for $r = 1$ to $r = 10$ values. For this reason, p values are accessible at inputs $r = 1$ to $r = 10$ only. For larger r inputs, the function returns the test statistic but not the p value and not the decision about H_0 at the significance level specified by the user.

3.3. **Function** `quantile_tables`**.** To access the test quantile tables for the partial sums of the Airy_1 point process, discussed in Section 2.3 and in Section 5, the user can call the `quantile_tables()` function. Quantile tables are available for $r = 1$ to $r = 10$. The function returns a numeric matrix, where the $0.ab$ quantile corresponds to the row $0.a$ and the column b. For example:

```
R> quantile_tables(r=1)
```

	0	1	2	3	4	5	6	7	8	9
0.0	-Inf	-3.90	-3.61	-3.43	-3.30	-3.18	-3.08	-3.00	-2.92	-2.85
0.1	-2.78	-2.72	-2.67	-2.61	-2.56	-2.51	-2.46	-2.41	-2.37	-2.33
0.2	-2.29	-2.24	-2.20	-2.17	-2.13	-2.09	-2.05	-2.02	-1.98	-1.95
0.3	-1.91	-1.88	-1.84	-1.81	-1.78	-1.74	-1.71	-1.68	-1.65	-1.62
0.4	-1.58	-1.55	-1.52	-1.49	-1.46	-1.43	-1.40	-1.36	-1.33	-1.30

```
0.5   -1.27  -1.24  -1.21  -1.17  -1.14  -1.11  -1.08  -1.05  -1.01  -0.98
0.6   -0.95  -0.91  -0.88  -0.85  -0.81  -0.78  -0.74  -0.71  -0.67  -0.63
0.7   -0.59  -0.56  -0.52  -0.48  -0.44  -0.39  -0.35  -0.31  -0.26  -0.22
0.8   -0.17  -0.12  -0.07  -0.01   0.04   0.10   0.16   0.23   0.30   0.37
0.9    0.45   0.53   0.63   0.73   0.85   0.98   1.14   1.33   1.60   2.02
```

3.4. **Function** `sim_function`**.** `sim_function()` is an auxiliary function that allows the user to calculate an empirical p value based on a simulation of the data generating process $\widehat{H}_0$ stated in of Bykhovskaya and Gorin [2025, equation (10)]; equivalently this is the model (1) based on mean 0 Gaussian errors ε_t and $\Gamma_1 = \cdots = \Gamma_{k-1} = 0$. This function should be used only for *quick approximate assessments*, as precise computation of the distribution of the test statistic requires a very large number of simulations, as discussed in Section 2.3.

```
sim_function(N, tau, stat_value, k = 1,
  r = 1, fin_sample_corr = FALSE, sim_num = 1000,  seed)
```

`N` The number of time series used in simulations.

`tau` The length of the time series used in simulations. If time is indexed as $t = 0, 1, \ldots, T$, then $\tau = T + 1$.

`stat_value` The test statistic value for which the p value is calculated.

`k` The number of lags that we wish to employ in the vector autoregression. The default value is $k = 1$.

`r` The number of largest eigenvalues used in the test. The default value is $r = 1$.

`fin_sample_corr` A boolean variable indicating whether we wish to employ finite sample correction on our test statistics. The default value is `fin_sample_corr=FALSE`.

`sim_num` The number of simulations that the function conducts for H_0. The default value is `sim_num = 1000`.

`seed` A numeric variable for the user to set to make simulations replicable. If not set by the user, there is no seed set for the simulations.

The function `sim_function()` runs the cointegration test (following the steps of Section 2) on simulated data generated under the evolution (1) based on mean 0 Gaussian errors ε_t and $\Gamma_1 = \cdots = \Gamma_{k-1} = 0$ and calculates the empirical p value based on the test statistic (7) corresponding to `r` specified by the user. The empirical p value is defined as the fraction of realizations larger than the specified `stat_value`. For comparison purposes, it is advised to specify the same parameters `k` and `r` as one expects to use for the run of `largevar()` for the desired data set.

`sim_function()` returns a list object that contains the simulation values, the empirical p value and a histogram of the distribution of simulated test statistic values (which is an approximation of the probability distribution of the test statistic).

4. Examples

This section provides two examples of the usage of the package. Section 4.1 replicates the S&P100 example from Bykhovskaya and Gorin [2022, 2025], while Section 4.2 uses simulated data. Both examples include the code, which can be copied into R.

4.1. **S&P100.** We use logarithms of weekly adjusted closing prices of assets in the S&P100 over ten years (01.01.2010–01.01.2020), which gives us $\tau = 522$ observations across time. The S&P100 includes 101 stocks, with Google having two classes of stocks. We use 92 of those stocks, those for which data were available for our chosen time period. Only one of Google's two listed stocks is kept in the sample. Therefore, $N = 92$, $T = 521$ and $T/N \approx 5.66$. We obtained the raw data from Yahoo! Finance and made the sample available in the "data" folder of the package for convenient data loading:

```
data("s_p100_price")
```

We first make necessary transformations, then convert to a numeric matrix to match function requirements:

```
dataSP <- log(s_p100_price[, seq(2, dim(s_p100_price)[2])])
dataSP <- as.matrix(dataSP)
```

There is documentation available for the following function which can be called using

```
‘?‘(largevar)
```

The following code conducts the cointegration test and displays its results:

```
result <- largevar(data = dataSP,
                   k = 1,
                   r = 1,
                   fin_sample_corr = FALSE,
                   plot_output = TRUE,
                   significance_level = 0.05)

result
```

Since we set `plot_output=TRUE`, we obtain a histogram of eigenvalues solving (6), as shown in Figure 1. The resemblance of the histogram with the theoretical curve is very good and we expect that our cointegration test should be applicable to this data set. The remaining output of `largevar()` is displayed in the console as:

```
Output for the largevar function
================================
Cointegration test for high-dimensional VAR(k)   T= 521 N= 92

 10% Crit. value 5% Crit. value 1% Crit. value Test stat.
            0.45           0.98           2.02      -0.28

If the test statistic is larger than the quantile, reject H0.
============================================================
Test statistic: -0.2777314
The p-value is  0.23
Decision about H0:  0
```

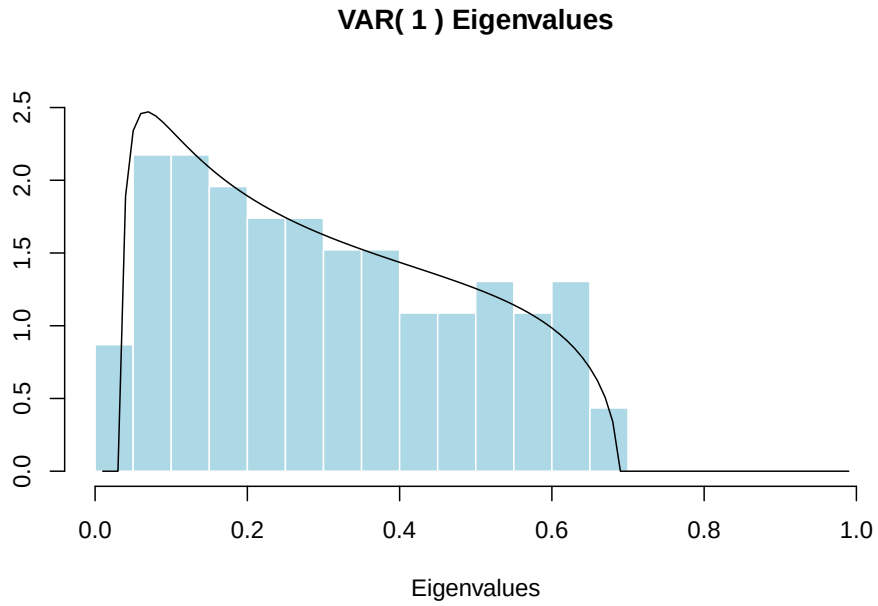

FIGURE 1. Histogram of eigenvalues and Wachter distribution, as discussed in Section 2.4.

The decision 0 means that we do not reject H_0, i.e., it is likely that the data set has no cointegration. If we want to individually access certain values from the output list, we can do it by referencing the elements of the list:

```
R> result$statistic
[1] -0.2777314
R> result$significance_test$p_value
[1] 0.23
R> result$significance_test$boolean_decision
[1] 0
R> result$significance_test$significance_table
        0.90    0.95    0.97    0.99   Test stat.
r=1     0.45    0.98    1.33    2.02   -0.2777314
r=2    -1.87   -1.09   -0.57    0.42   -1.4995879
r=3    -5.90   -4.90   -4.24   -2.99   -5.4154889
r=4   -11.35  -10.15   -9.37   -7.87  -10.5527603
r=5   -18.07  -16.69  -15.79  -14.07  -16.7460847
r=6   -25.95  -24.40  -23.38  -21.45  -23.2178976
r=7   -34.90  -33.19  -32.07  -29.95  -31.1080001
r=8   -44.88  -43.01  -41.79  -39.47  -39.3197363
r=9   -55.82  -53.80  -52.48  -49.99  -49.8419822
r=10  -67.70  -65.53  -64.12  -61.45  -60.4894485
```

We can further compare the exact p value (which was outputted by `largevar()`) with a sample value obtained trough 1000 simulations, by running `sim_function()`:

```
result2 <- sim_function(N = 92,
                        tau = 522,
                        stat_value = result$statistic,
                        k = 1,
                        r = 1,
                        fin_sample_corr = FALSE,
                        sim_num = 1000,
                        seed = 333)
> result2

Output for the sim_function function
====================================
The empirical p-value is  0.247
```

As we see, the empirical p value 0.247 is close to the previous output 0.23, but there is a small mismatch, as expected.

4.2. **Simulation example.** We also present an example based on simulated data that users can replicate. The code below generates VAR(2) with $N = 100$, $T = 1500$, and

$$\begin{pmatrix}\Delta X_{1t}\\ \Delta X_{2t}\end{pmatrix} = \begin{pmatrix}-0.9 & 0.8\\ 0 & 0\end{pmatrix}\begin{pmatrix}X_{1t-2}\\ X_{2t-2}\end{pmatrix} + \begin{pmatrix}-0.7 & 0.8\\ 0 & 0.3\end{pmatrix}\begin{pmatrix}\Delta X_{1t-1}\\ \Delta X_{2t-1}\end{pmatrix} + \begin{pmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{pmatrix},\ t = 1,\dots,T,$$

$$\begin{pmatrix}\Delta X_{4t}\\ \Delta X_{5t}\end{pmatrix} = \begin{pmatrix}-0.9 & 0.8\\ 0 & 0\end{pmatrix}\begin{pmatrix}X_{4t-2}\\ X_{5t-2}\end{pmatrix} + \begin{pmatrix}-1.2 & 0.8\\ 0 & 0.25\end{pmatrix}\begin{pmatrix}\Delta X_{4t-1}\\ \Delta X_{5t-1}\end{pmatrix} + \begin{pmatrix}\varepsilon_{4t}\\ \varepsilon_{5t}\end{pmatrix},\ t = 1,\dots,T, \tag{15}$$

$$\Delta X_{it} = \varepsilon_{it},\ i \neq 1,2,4,5,\ t = 1,\dots,T,$$

where $\Delta X_{it} := X_{it} - X_{it-1}$. The process is initialized by vectors X_0, X_{-1} with independent standard normal coordinates. The data generating process (15) corresponds to a matrix Π of rank 2: Π has two nonzero and linearly independent rows. To be more precise, the coefficient matrices in Eq. (15) correspond to $N-2$ unit root and 2 stationary components. We create a data set based on evolution (15) and Gaussian errors, and then run the cointegration test on it, as follows.

The following code constructs matrices Π and Γ that by construction create two separate cointegrating systems, the first and second, and the fourth and fifth time series:

```
set.seed(333)
T_ <- 1500
N <- 100

Pi <- matrix(0, N, N)
Pi[1:5, 1:5] <- matrix(c(-0.9, rep(0, 4), 0.8, rep(0, 12), -0.9,
                         rep(0, 4), 0.8, 0), 5, 5)
Gamma <- matrix(0, N, N)
Gamma[1:5, 1:5] <- matrix(c(-0.7, rep(0, 4), 0.8, 0.3, rep(0, 11), -1.2,
                            rep(0, 4), 0.8, 0.25), 5, 5)
```

The initialization of the time series by setting all the below values:

```
Xminus1 <- matrix(rnorm(N), N, 1)
X0 <- matrix(rnorm(N), N, 1)
dX <- matrix(0, N, T_)
```

```
dX0 <- X0 - Xminus1

epsilon <- matrix(rnorm(N * T_), N, T_)
dX[ , 1] <- Pi %*% Xminus1 + Gamma %*% dX0 + epsilon[ , 1]
dX[ , 2] <- Pi %*% X0 + Gamma %*% dX[ , 1] + epsilon[ , 2]
dX[ , 3] <- Pi %*% (X0 + dX[ , 1]) + Gamma %*% dX[ , 2] + epsilon[ , 3]
```

The development of the system up to T is calculated, starting with changes dX for each t:

```
for (t in 4:T_) {
  dX[ , t] <- Pi %*% (X0 + rowSums(dX[ , 1:(t - 2)])) +
              Gamma %*% dX[ , t - 1] + epsilon[ , t]
}

data_sim <- matrix(0, N, T_ + 1)
data_sim[ , 1] <- X0
for (t in 2:(T_ + 1)) {
  data_sim[ , t] <- data_sim[ , t - 1] + dX[ , t - 1]
}
data_sim <- t(data_sim)
```

Finally, we conduct the cointegration test and display the results:

```
result <- largevar(data = data_sim, k = 2, r = 2, fin_sample_corr = FALSE,
                         plot_output = TRUE, significance_level = 0.05)
> result

Output for the largevar function
================================
Cointegration test for high-dimensional VAR(k)   T= 1500 N= 100

 10% Crit. value 5% Crit. value 1% Crit. value Test stat.
           -1.87          -1.09           0.42      48.43

If the test statistic is larger than the quantile, reject H0.
=============================================================
Test statistic: 48.42677
The p-value is  0.01
Decision about H0:  1
```

Since the decision is 1, we reject H_0 and conclude that the data set has cointegration. The rejection is in line with the largest eigenvalue being significantly to the right from λ_+. The separation of eigenvalues is clearly visible in the histogram output, see Figure 2.

If we want to take a look at how the significance of our test statistics vary across different choices of r, we can call the table below. The p values for our test statistics stay below 0.01.

```
R> result$significance_test$significance_table
```

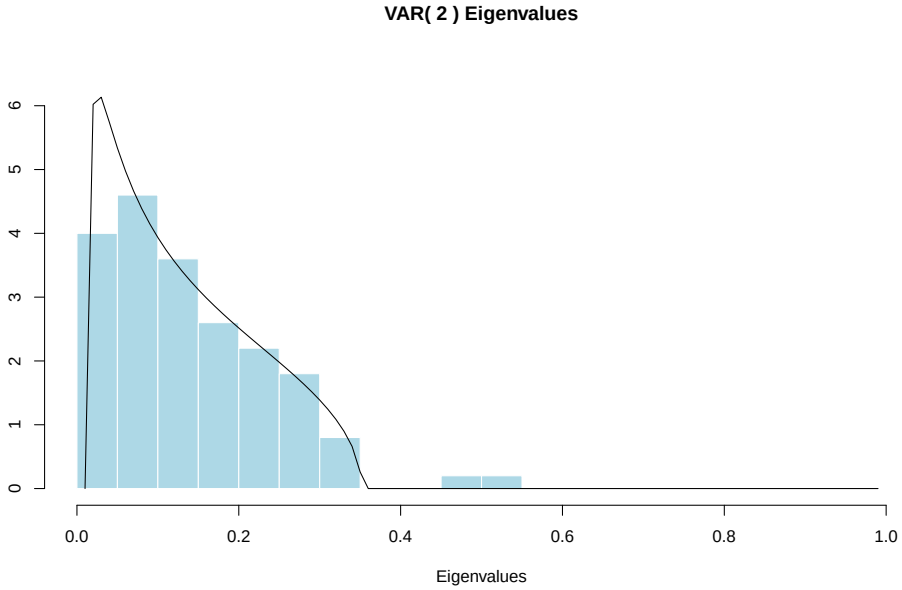

FIGURE 2. Histogram of eigenvalues and Wachter distribution, as discussed in Section 2.4.

	0.90	0.95	0.99	Test stat.
r=1	0.45	0.98	2.02	27.357695
r=2	-1.87	-1.09	0.42	48.426766
r=3	-5.90	-4.90	-2.99	46.505972
r=4	-11.35	-10.15	-7.87	44.057939
r=5	-18.07	-16.69	-14.07	39.016668
r=6	-25.95	-24.40	-21.45	31.463442
r=7	-34.90	-33.19	-29.95	22.644198
r=8	-44.88	-43.01	-39.47	12.781779
r=9	-55.82	-53.80	-49.99	2.638057
r=10	-67.70	-65.53	-61.45	-7.878603

5. TABLES OF QUANTILES

We include the tables discussed in Section 2.3, both inside the package and in this section. These tables are used inside `largevar()` to obtain the quantiles. The tables below present our simulation results. The $0.ab$ quantile in each table corresponds to the row $0.a$ and the column b. The standard deviations of our results suggest that the error is at most ± 1 in the third digit of the elements of the Airy_1 sequence, meaning that the error is ± 0.01 for $r = 1$ and ± 0.1 for $r = 10$.

q	0	1	2	3	4	5	6	7	8	9
0.0	$-\infty$	-3.90	-3.61	-3.43	-3.30	-3.18	-3.08	-3.00	-2.92	-2.85
0.1	-2.78	-2.72	-2.67	-2.61	-2.56	-2.51	-2.46	-2.41	-2.37	-2.33
0.2	-2.29	-2.24	-2.20	-2.17	-2.13	-2.09	-2.05	-2.02	-1.98	-1.95
0.3	-1.91	-1.88	-1.84	-1.81	-1.78	-1.74	-1.71	-1.68	-1.65	-1.62
0.4	-1.58	-1.55	-1.52	-1.49	-1.46	-1.43	-1.40	-1.36	-1.33	-1.30
0.5	-1.27	-1.24	-1.21	-1.17	-1.14	-1.11	-1.08	-1.05	-1.01	-0.98
0.6	-0.95	-0.91	-0.88	-0.85	-0.81	-0.78	-0.74	-0.71	-0.67	-0.63
0.7	-0.59	-0.56	-0.52	-0.48	-0.44	-0.39	-0.35	-0.31	-0.26	-0.22
0.8	-0.17	-0.12	-0.07	-0.01	0.04	0.10	0.16	0.23	0.30	0.37
0.9	0.45	0.53	0.63	0.73	0.85	0.98	1.14	1.33	1.60	2.02

TABLE 1. Quantiles of $\mathfrak{a}_1$ based on 10^7 Monte Carlo simulations of $10^8 \times 10^8$ tridiagonal matrices.

q	0	1	2	3	4	5	6	7	8	9
0.0	$-\infty$	-8.93	-8.44	-8.12	-7.88	-7.69	-7.52	-7.37	-7.24	-7.12
0.1	-7.01	-6.91	-6.81	-6.72	-6.63	-6.54	-6.46	-6.39	-6.31	-6.24
0.2	-6.17	-6.10	-6.04	-5.97	-5.91	-5.85	-5.79	-5.73	-5.67	-5.61
0.3	-5.56	-5.50	-5.45	-5.39	-5.34	-5.29	-5.23	-5.18	-5.13	-5.08
0.4	-5.03	-4.97	-4.92	-4.87	-4.82	-4.77	-4.72	-4.67	-4.62	-4.57
0.5	-4.52	-4.47	-4.42	-4.37	-4.32	-4.27	-4.22	-4.17	-4.11	-4.06
0.6	-4.01	-3.96	-3.91	-3.85	-3.80	-3.74	-3.69	-3.63	-3.57	-3.52
0.7	-3.46	-3.40	-3.34	-3.27	-3.21	-3.15	-3.08	-3.01	-2.94	-2.87
0.8	-2.80	-2.72	-2.65	-2.57	-2.48	-2.39	-2.30	-2.20	-2.10	-1.99
0.9	-1.87	-1.75	-1.61	-1.46	-1.29	-1.09	-0.86	-0.57	-0.19	0.42

TABLE 2. Quantiles of $\mathfrak{a}_1 + \mathfrak{a}_2$ based on 10^7 Monte Carlo simulations of $10^8 \times 10^8$ tridiagonal matrices.

q	0	1	2	3	4	5	6	7	8	9
0.0	$-\infty$	-15.2	-14.6	-14.1	-13.8	-13.5	-13.3	-13.1	-12.9	-12.8
0.1	-12.6	-12.5	-12.4	-12.2	-12.1	-12.0	-11.9	-11.8	-11.7	-11.6
0.2	-11.5	-11.4	-11.3	-11.3	-11.2	-11.1	-11.0	-10.9	-10.9	-10.8
0.3	-10.7	-10.6	-10.6	-10.5	-10.4	-10.3	-10.3	-10.2	-10.1	-10.1
0.4	-10.0	-9.93	-9.87	-9.80	-9.73	-9.67	-9.60	-9.54	-9.47	-9.40
0.5	-9.34	-9.27	-9.21	-9.14	-9.07	-9.01	-8.94	-8.87	-8.80	-8.74
0.6	-8.67	-8.60	-8.53	-8.46	-8.39	-8.32	-8.25	-8.17	-8.10	-8.02
0.7	-7.95	-7.87	-7.79	-7.71	-7.63	-7.55	-7.46	-7.37	-7.28	-7.19
0.8	-7.10	-7.00	-6.90	-6.79	-6.68	-6.57	-6.45	-6.33	-6.19	-6.05
0.9	-5.90	-5.74	-5.56	-5.37	-5.15	-4.90	-4.60	-4.24	-3.76	-2.99

TABLE 3. Quantiles of $\sum_{i=1}^{3} \mathfrak{a}_i$ based on 10^7 Monte Carlo simulations of $10^8 \times 10^8$ tridiagonal matrices.

q	0	1	2	3	4	5	6	7	8	9
0.0	$-\infty$	-22.7	-21.9	-21.4	-21.0	-20.7	-20.4	-20.1	-19.9	-19.7
0.1	-19.5	-19.3	-19.2	-19.0	-18.9	-18.8	-18.6	-18.5	-18.4	-18.3
0.2	-18.2	-18.0	-17.9	-17.8	-17.7	-17.6	-17.5	-17.4	-17.3	-17.3
0.3	-17.2	-17.1	-17.0	-16.9	-16.8	-16.7	-16.6	-16.6	-16.5	-16.4
0.4	-16.3	-16.2	-16.1	-16.1	-16.0	-15.9	-15.8	-15.7	-15.7	-15.6
0.5	-15.5	-15.4	-15.3	-15.3	-15.2	-15.1	-15.0	-14.9	-14.8	-14.8
0.6	-14.7	-14.6	-14.5	-14.4	-14.4	-14.3	-14.2	-14.1	-14.0	-13.9
0.7	-13.8	-13.7	-13.6	-13.5	-13.4	-13.3	-13.2	-13.1	-13.0	-12.9
0.8	-12.8	-12.7	-12.6	-12.4	-12.3	-12.2	-12.0	-11.9	-11.7	-11.5
0.9	-11.4	-11.2	-11.0	-10.7	-10.5	-10.2	-9.80	-9.37	-8.79	-7.87

TABLE 4. Quantiles of $\sum_{i=1}^{4} \mathfrak{a}_i$ based on 10^7 Monte Carlo simulations of $10^8 \times 10^8$ tridiagonal matrices.

q	0	1	2	3	4	5	6	7	8	9
0.0	$-\infty$	-31.3	-30.3	-29.7	-29.2	-28.9	-28.5	-28.2	-28.0	-27.8
0.1	-27.5	-27.4	-27.2	-27.0	-26.8	-26.7	-26.5	-26.4	-26.2	-26.1
0.2	-26.0	-25.8	-25.7	-25.6	-25.5	-25.3	-25.2	-25.1	-25.0	-24.9
0.3	-24.8	-24.7	-24.6	-24.5	-24.4	-24.3	-24.2	-24.1	-24.0	-23.9
0.4	-23.8	-23.7	-23.6	-23.5	-23.4	-23.3	-23.2	-23.1	-23.1	-23.0
0.5	-22.9	-22.8	-22.7	-22.6	-22.5	-22.4	-22.3	-22.2	-22.1	-22.0
0.6	-21.9	-21.8	-21.7	-21.6	-21.5	-21.4	-21.3	-21.2	-21.1	-21.0
0.7	-20.9	-20.8	-20.7	-20.6	-20.5	-20.4	-20.2	-20.1	-20.0	-19.9
0.8	-19.7	-19.6	-19.5	-19.3	-19.2	-19.0	-18.8	-18.7	-18.5	-18.3
0.9	-18.1	-17.9	-17.6	-17.3	-17.0	-16.7	-16.3	-15.8	-15.1	-14.1

TABLE 5. Quantiles of $\sum_{i=1}^{5} \mathfrak{a}_i$ based on 10^7 Monte Carlo simulations of $10^8 \times 10^8$ tridiagonal matrices.

q	0	1	2	3	4	5	6	7	8	9
0.0	$-\infty$	-40.9	-39.8	-39.1	-38.6	-38.1	-37.8	-37.4	-37.2	-36.9
0.1	-36.7	-36.4	-36.2	-36.0	-35.8	-35.6	-35.5	-35.3	-35.1	-35.0
0.2	-34.8	-34.7	-34.6	-34.4	-34.3	-34.2	-34.0	-33.9	-33.8	-33.7
0.3	-33.5	-33.4	-33.3	-33.2	-33.1	-33.0	-32.9	-32.7	-32.6	-32.5
0.4	-32.4	-32.3	-32.2	-32.1	-32.0	-31.9	-31.8	-31.7	-31.6	-31.5
0.5	-31.4	-31.3	-31.1	-31.0	-30.9	-30.8	-30.7	-30.6	-30.5	-30.4
0.6	-30.3	-30.2	-30.1	-30.0	-29.9	-29.7	-29.6	-29.5	-29.4	-29.3
0.7	-29.1	-29.0	-28.9	-28.8	-28.7	-28.5	-28.4	-28.3	-28.1	-28.0
0.8	-27.8	-27.7	-27.5	-27.3	-27.2	-27.0	-26.8	-26.6	-26.4	-26.2
0.9	-29.0	-25.7	-25.4	-25.1	-24.8	-24.4	-23.9	-23.4	-22.6	-21.5

TABLE 6. Quantiles of $\sum_{i=1}^{6} \mathfrak{a}_i$ based on 10^7 Monte Carlo simulations of $10^8 \times 10^8$ tridiagonal matrices.

q	0	1	2	3	4	5	6	7	8	9
0.0	$-\infty$	-51.5	-50.3	-49.5	-48.9	-48.4	-48.0	-47.6	-47.3	-47.0
0.1	-46.8	-46.5	-46.3	-46.1	-45.8	-45.6	-45.5	-45.3	-45.1	-44.9
0.2	-44.8	-44.6	-44.4	-44.3	-44.1	-44.0	-43.9	-43.7	-43.6	-43.4
0.3	-43.3	-43.2	-43.0	-42.9	-42.8	-42.7	-42.5	-42.4	-42.3	-42.2
0.4	-42.1	-41.9	-41.8	-41.7	-41.6	-41.5	-41.4	-41.2	-41.1	-41.0
0.5	-40.9	-40.8	-40.7	-40.5	-40.4	-40.3	-40.2	-40.1	-40.0	-39.8
0.6	-39.7	-39.6	-39.5	-39.4	-39.2	-39.1	-39.0	-38.8	-38.7	-38.6
0.7	-38.5	-38.3	-38.2	-38.0	-37.9	-37.8	-37.6	-37.5	-37.3	-37.3
0.8	-37.0	-36.8	-36.6	-36.4	-36.3	-36.1	-35.9	-35.6	-35.4	-35.2
0.9	-34.9	-34.6	-34.3	-34.0	-33.6	-33.2	-32.7	-32.1	-31.2	-30.0

TABLE 7. Quantiles of $\sum_{i=1}^{7} \mathfrak{a}_i$ based on 10^7 Monte Carlo simulations of $10^8 \times 10^8$ tridiagonal matrices.

q	0	1	2	3	4	5	6	7	8	9
0.0	$-\infty$	-63.0	-61.7	-60.8	-60.2	-59.7	-59.2	-58.8	-58.5	-58.1
0.1	-57.8	-57.6	-57.3	-57.1	-56.8	-56.6	-56.4	-56.2	-56.0	-55.8
0.2	-55.6	-55.5	-55.3	-55.1	-55.0	-54.8	-54.7	-54.5	-54.4	-54.2
0.3	-54.1	-53.9	-53.8	-53.6	-53.5	-53.4	-53.2	-53.1	-53.0	-52.8
0.4	-52.7	-52.6	-52.4	-52.3	-52.2	-52.0	-51.9	-51.8	-51.7	-51.5
0.5	-51.4	-51.3	-51.2	-51.0	-50.9	-50.8	-50.6	-50.5	-50.4	-50.3
0.6	-50.1	-50.0	-49.9	-49.7	-49.6	-49.5	-49.3	-49.2	-49.0	-48.9
0.7	-48.8	-48.6	-48.5	-48.3	-48.1	-48.0	-47.8	-47.7	-47.5	-47.3
0.8	-47.1	-47.0	-46.8	-46.6	-46.4	-46.1	-45.9	-45.7	-45.4	-45.2
0.9	-44.9	-44.6	-44.2	-43.9	-43.5	-43.0	-42.5	-41.8	-40.9	-39.5

TABLE 8. Quantiles of $\sum_{i=1}^{8} \mathfrak{a}_i$ based on 10^7 Monte Carlo simulations of $10^8 \times 10^8$ tridiagonal matrices.

q	0	1	2	3	4	5	6	7	8	9
0.0	$-\infty$	-75.5	-74.0	-73.1	-72.4	-71.8	-71.3	-70.9	-70.5	-70.2
0.1	-69.9	-69.6	-69.3	-69.0	-68.8	-68.5	-68.3	-68.1	-67.9	-67.7
0.2	-67.5	-67.3	-67.1	-66.9	-66.7	-66.6	-66.4	-66.2	-66.1	-65.9
0.3	-65.7	-65.6	-65.4	-65.3	-65.1	-65.0	-64.8	-64.7	-64.6	-64.4
0.4	-64.3	-64.1	-64.0	-63.8	-63.7	-63.6	-63.4	-63.3	-63.2	-63.0
0.5	-62.9	-62.7	-62.6	-62.5	-62.3	-62.2	-62.1	-61.9	-61.8	-61.6
0.6	-61.5	-61.4	-61.2	-61.1	-60.9	-60.8	-60.6	-60.5	-60.3	-60.2
0.7	-60.0	-59.9	-59.7	-59.5	-59.4	-59.2	-59.0	-58.8	-58.6	-58.5
0.8	-58.3	-58.1	-57.9	-57.6	-57.4	-57.2	-56.9	-56.7	-56.4	-56.1
0.9	-55.8	-55.5	-55.1	-54.7	-54.3	-53.8	-53.2	-52.5	-51.5	-50.0

TABLE 9. Quantiles of $\sum_{i=1}^{9} \mathfrak{a}_i$ based on 10^7 Monte Carlo simulations of $10^8 \times 10^8$ tridiagonal matrices.

q	0	1	2	3	4	5	6	7	8	9
0.0	$-\infty$	-88.8	-87.2	-86.2	-85.5	-84.9	-84.3	-83.9	-83.5	-83.1
0.1	-82.8	-82.4	-82.1	-81.8	-81.6	-81.3	-81.1	-80.8	-80.6	-80.4
0.2	-80.2	-80.0	-79.8	-79.6	-79.4	-79.2	-79.0	-78.9	-78.7	-78.5
0.3	-78.3	-78.2	-78.0	-77.8	-77.7	-77.5	-77.4	-77.2	-77.1	-76.9
0.4	-76.8	-76.6	-76.5	-76.3	-76.2	-76.0	-75.9	-75.7	-75.6	-75.4
0.5	-75.3	-75.1	-75.0	-74.8	-74.7	-74.5	-74.4	-74.2	-74.1	-73.9
0.6	-73.8	-73.6	-73.5	-73.3	-73.2	-73.0	-72.8	-72.7	-72.5	-72.4
0.7	-72.2	-72.0	-71.8	-71.7	-71.5	-71.3	-71.1	-70.9	-70.7	-70.5
0.8	-70.3	-70.1	-69.9	-69.6	-69.4	-69.2	-68.9	-68.6	-68.3	-68.0
0.9	-67.7	-67.3	-67.0	-66.5	-66.1	-65.5	-64.9	-64.1	-63.1	-61.5

TABLE 10. Quantiles of $\sum_{i=1}^{10} \mathfrak{a}_i$based on 10^7 Monte Carlo simulations of $10^8 \times 10^8$ tridiagonal matrices.

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